Varianta 045

Subiectul I

a) $\sqrt{2}$. b) $\frac{19}{\sqrt{3}}$. c) x-6y=20. d) c=$\frac{\pi}{2}$. e) 6. f) a=-2.

Subiectul II

1. a) 0. b) $\frac{1}{3}$. c) 2. d) 1. e) $\begin{pmatrix} 7 & -2 \\ -3 & 1 \end{pmatrix}$.

2. a) e^x-1. b) $e-\frac{5}{2}$. c) $f''(x)>0$. d) e-1. e) $f(x)\geq 0$. $(\forall)x\in$ **R**.

Subiectul III

a) $f_n(k)=\frac{k(k-1)...(k-n+1)}{n!}=C_k^n$. $k\geq n$.

b) Pentru $k\geq n$, $f_n(k)=C_k^n \in$ **Z**.

Pentru $k\in\{0.1.\dots.n-1\}$ $f_n(k)=0\in$ **Z**.

Pentru k<0. $f_n(k)=\frac{(-1)^n(-k)(-k+1)...(-k+n-1)}{n!}=(-1)^n C^n_{-k+n-1}\in$ **Z**.

c) Luam $g=f_3=\frac{x(x-1)(x-2)}{6}=\frac{1}{6}x^3-\frac{1}{2}x^2+\frac{1}{3}x$ care are toti coeficientii neintregi, iar $f_3(k)\in$ **Z**. $(\forall)k\in$ **Z**.

d) Cum f_n are n factori de gradul $1\Rightarrow$ grad f_n=n.

e) Demonstram ca exista a_0, a_1, a_2, $a_3\in$ **C** unice a.i $h=a_0\cdot 1+a_1x+a_2\frac{x(x-1)}{2}+a_3\frac{x(x-1)(x-2)}{6}$

x=0. $h(0)=a_0$.

x=1. $h(1)=a_0+a_1$. deci $a_1=h(1)-h(0)$.

x=2. $a_2=h(2)-2h(1)+h(0)$.

x=3. $a_3=h(3)-a_0-3a_1$.

Astfel am demonstrat ca a_0, a_1, a_2, a_3 sunt determinate in mod unic.

f) Fie $w=a_0f_0+a_1f_1+a_2f_2+a_3f_3$ conf lui e) luam $w(0)=a_0\in$ **Z**.

$w(1)=a_0+a_1\in$ **Z**. $w(2)=a_0+2a_1+a_2\in$ **Z**.

$w(3)=a_0+3a_1+3a_2+a_3\in$ **Z** rezulta ca a_0, a_1, a_2,$a_3\in$ **Z**.

De aici si din $f_n(k)\in$ **Z**. $(\forall)k\in$ **Z** $\Rightarrow w(k)\in$ **Z**.

g) Din e $\Rightarrow u=a_0f_0+a_1f_1+a_2f_2+a_3f_3$ cu $a_i\in$ **Z** $\Rightarrow u(k)\in$ **Z**. $(\forall)k\in$ **Z**

avem: $u=b_3x^3+b_2x^2+b_1x+b_0$, $b_i\in$ **Q**

Analizam cazul $b_3>0$(cazul $b_3<0$ este similar)

Avem $\lim\limits_{x\to\infty}u(x)=-\infty, \lim\limits_{x\to\infty}u(x)=\infty$ si exista $A\in$ **R** astfel ca $u(x)\leq u(A), \forall x\leq A$ si u este strict crescatoare pe $[A,\infty)$

Polinomul $v(x)=u(x+1)-u(x)$ are gradul doi si $\lim\limits_{x\to\infty}u(x+1)-u(x)=\infty$

Exista $B>A$ astfel ca $u(x+1)-u(x)\geq 2, \forall x\geq B$. Daca alegem $M\in$ **N**, $M>B$ si $u(M+1)\geq M+2$ deci $u(k)\neq M+1, \forall k\in$ **Z**

Subiectul IV
a) se verifica usor.

b) $f_2(x)=\int_0^x t-\sin t\,dt=\frac{x^2}{2}+\cos x-1$.

c) Fie $a_n=\frac{x^n}{n!}$. $(\forall)$ $n\in N^*$, avem $a_n>0$ $(\forall)n\in \mathbf{N}^*$ si $\lim\limits_{x\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{x\to\infty}\frac{x}{n+1}=0$.

Conform criteriului raportului avem $\lim\limits_{n\to\infty}a_n=0$.

d) Avem $\lim\limits_{x\to\infty}f_1=\infty$ si $\lim\limits_{x\to\infty}\frac{f(x)}{x}=1$. apoi $f_1(x)-x=-\sin x$ $(\forall)$ $x\in\mathbf{R}$ cum nu exista $\lim\limits_{x\to\infty}(-\sin x)$, graficul functiei f_1 nu are asimptota spre $+\infty$.

e) Cum verificarea este facuta, ramane sa aratam ca $P(n)\Rightarrow P(n+1)$

$f_{(2n+1)}(x)=\int_0^x f_{2n}(t)dt=\frac{x^{2n+1}}{(2n+1)!}-\frac{x^{2n-1}}{(2n-1)!}+\ldots+(-1)^{n-1}\frac{x^3}{3!}+(-1)^n x+(-1)^{n+1}\sin x$.

$f_{(2n+2)}(x)=\int_0^x f_{2n}(t)dt=\frac{x^{2n+2}}{(2n+2)!}-\frac{x^{2n}}{2n!}+\ldots+(-1)^{n-1}\frac{x^4}{4!}+(-1)^n\frac{x^2}{2!}+(-1)^{n+1}+(-1)^{n+2}\cos x$

f) Aratam inductiv ca $0\le f_n(x)\le 2\frac{x^n}{n!}$ $(\forall)n\in\mathbf{N}^*$, $x>0$ at $n=1$ din $0\le f_0(t)\le 2$ $(\forall)$ $t\in\mathbf{R}\Rightarrow$ integrand ca $0\le f_1(x)\le 2x$. $(\forall)x>0$ apoi din $0\le f_n(t)\le 2\frac{t^n}{n!}$ integrand$\Rightarrow$ $0\le f_{n+1}(x)\le 2\frac{x^{n+1}}{(n+1)!}$. $(\forall)x>0$

g) din c) si f) $\Rightarrow$ ca $\lim\limits_{x\to\infty}f_n(x)=0$, $(\forall)x>0$ si $\lim\limits_{x\to\infty}(-1)^k f_n(x)=0$ $(\forall)$ $x>0$

Fie $b_n(x)=1-\frac{x^2}{2!}+\ldots+(-1)^n\frac{x^{2n}}{(2n)!}$. Din f) $\Rightarrow\lim\limits_{x\to\infty}b_n(x)=\cos x$. $(\forall)x>0$.Deoarece funcțiile b_n şi cos sunt funcții impare, rezultă egalitatea şi pentru $x\le 0$

Asadar $\lim\limits_{n\to\infty}b_n(x)=\cos x$, $(\forall)$ $x\in\mathbf{R}$.